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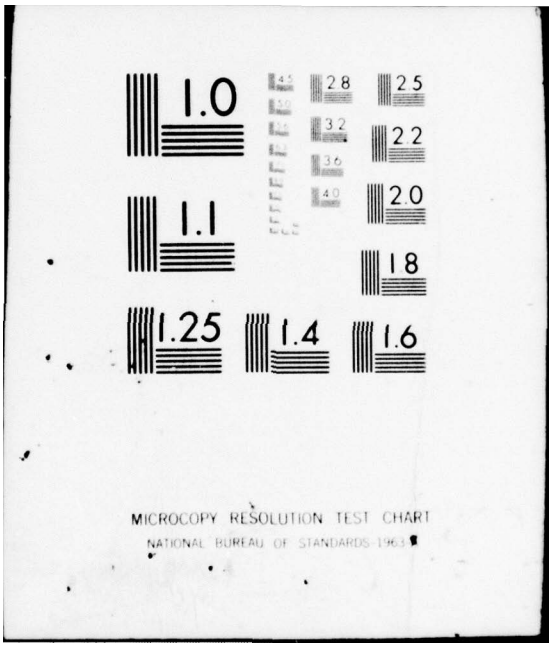
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A NOTE ON THE OPERATOR COMPACT IMPLICIT METHOD FOR THE WAVE EQUATION

BY
Melvyn Ciment
Stephen H. Leventhal

16 NOVEMBER 1976

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In a previous paper a fourth order compact implicit scheme was presented for the second order wave equation. A very efficient factorization technique was developed when only second order terms were present. In this note we implement the operator compact implicit spatial discretization method for the second order wave equation when first order terms are present. The resulting algorithm is completely analogous to the compact implicit → next page

cont

algorithm when lower order terms were not present. For this more general operator compact implicit spatial approximation the same factorization as in our previous paper is developed.

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A Note on the Operator Compact Implicit Method for the Wave Equation

The results of this report represent a continuation of an effort to develop high accuracy efficient methods for solving hyperbolic equations. Here an efficient fourth order method for the multi-dimensional wave equation is presented.

These techniques are also being used to solve parabolic equations arising in viscous fluid flow problems for NAVAIR.

This study has been supported by the Naval Surface Weapons Center Independent Research Fund.

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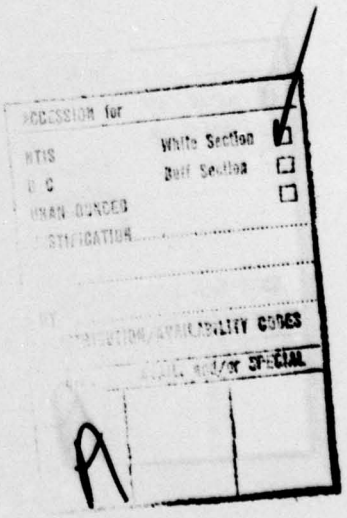

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INTRODUCTION

In a previous paper [1] a factorization technique which utilized compact implicit spatial and temporal approximations to the second order wave equation was developed. The method proceeded by separately implementing the so called compact implicit fourth order approximations for each of the individual derivative terms. Notwithstanding, the implicit nature of the basic approximation involved, a factorization technique was described which allowed one to resolve higher space dimension problems by requiring merely the solution of tridiagonal equations. In Section VI of [1] we observed a peculiar aspect of our factorization approach. The same approach required twice as much work when mixed order (first and second) spatial derivative terms were present.

Upon further examination of this approach it became apparent that the problem was numerically improperly posed in requiring too much additional data to complete the factorization.

In this note, we observe that by changing the underlying spatial approximation when lower order terms are present it is possible to obtain an algorithm which completely resembles our algorithm for the case when no lower order terms are present.

In a future paper [2] the operator compact implicit method is developed, in much the same way as here, for parabolic problems.

SPATIAL DISCRETIZATIONS

The classical finite difference approach for solving two point boundary value problems of the form

$$(2.1) \qquad L(u) = au_{xx} + bu_x = f, \quad x\in[0,1]$$

with $u(0)$, $u(1)$ given is to separately substitute standard approximations for the first and second derivatives in (2.1) and then solve the resulting system of equations. Accordingly, the fourth order compact implicit scheme applied to the solution of (2.1), requires that one solve

$$(2.2) \qquad L_h[U_j] = a_j(I + \frac{1}{12}\delta_x^2)^{-1}\frac{\delta_x^2}{h^2}U_j + b_j(I + \frac{1}{6}\delta_x^2)^{-1}\frac{\delta_o^x}{2h}U_j = f_j$$

[1] M. Ciment, S. H. Leventhal, "Higher Order Compact Implicit Schemes for the Wave Equation," Math. of Comp. Vol. 29, No. 132, 1975, pp. 985-994.

[2] M. Ciment, S. H. Leventhal and B. C. Weinberg, "The Operator Compact Implicit Method for Parabolic Equations," to appear.

The notation here picks up from the notation of [1]. References to an equation in [1] will be superscripted with an asterisk. The appearance of two implicit matrices Q_x, R_x (see (2.2)*, (6.2)*) "trapped inside the a_j, b_j" creates problems in trying to solve the resulting linear system of equations. Several people have tried block methods when these basic terms appear in problems [3]. A suitable tridiagonal relationship however can be obtained by merely abandoning attempts to represent the separate derivative terms. The approach we adopt is to represent $L(U_j)$ on three adjacent points up to highest order accuracy possible in relation to U_j on the same three points.

A Taylor series analysis shows that for L(u) on a uniform grid fourth order accuracy can be obtained by

$$q_j^+ L(u)_{j+1} + q_j^0 L(u)_j + q_j^- L(u)_{j-1} = \frac{r_j^+ u_{j+1} + r_j^0 u_j + r_j^- u_{j-1}}{h^2} \tag{2.3a}$$

where

$$\begin{cases} q_j^+ = 6a_j a_{j-1} + h(5a_{j-1} b_j - 2a_j b_{j-1}) - h^2 b_j b_{j-1} \\ q_j^0 = 4[15a_{j+1} a_{j-1} - 4h(a_{j+1} b_{j-1} - b_{j+1} a_{j-1}) - h^2 b_{j+1} b_{j-1}] \\ q_j^- = 6a_j a_{j+1} - h(5a_{j+1} b_j - 2a_j b_{j+1}) - h^2 b_j b_{j+1} \end{cases} \tag{2.3b}$$

$$\begin{cases} r_j^+ = \frac{1}{2}[q_j^+(2a_{j+1} + 3h\, b_{j+1}) + q_j^0(2a_j + hb_j) + q_j^-(2a_{j-1} - hb_{j-1})] \\ r_j^- = \frac{1}{2}[q_j^+(2a_{j+1} + hb_{j+1}) + q_j^0(2a_j - hb_j) + q_j^-(2a_{j-1} - 3hb_{j-1})] \\ r_j = -\frac{1}{2}[r_j^+ + r_j^-] \end{cases} \tag{2.3c}$$

The above relationship can be expressed in an operator form by defining tridiagonal displacement operators Q_x, R_x so that the equation (2.3) is represented by

$$Q_x L(U)_j = \frac{1}{h^2} R_x U_j \tag{2.4}$$

Note well, for conciseness of notation we are using Q_x here to represent a different operator from what Q_x represented in [1]. (Indeed in the case that

[3] R. Hirsh, "Higher Order Accurate Difference Solution of Fluid Mechanics Problems by a Compact Differencing Technique," J. Comp. Physics, 19, 1975, pp. 90-109.

$a \equiv 1$, $b \equiv 0$ then both are identical.) Below, however, we will still retain the definition of $Q_t = (I + \frac{1}{12}\delta_t^2)$ from our previous paper. The above relationships (2.3) were first presented by Swartz [4]. The operator compact implicit (OCI) representation $Q_x^{-1} R_x$ for $L(u)$ presents the same formal appearance as $Q_x^{-1} \frac{\delta_x^2}{h^2}$ did for u_{xx} alone. By noting this formal similarity it is clear that the OCI spatial discretization can be fully implemented into our factorization algorithm for the time dependent second order wave equation. This all depends on the underlying invertibility of Q_x. As Swartz observes, for the case that a and b are constants, Q_x is invertible so long as the so called mesh Reynolds number r satisfies

$$(2.5) \qquad |r| \equiv \left|\frac{hb}{a}\right| \leq \sqrt{12}$$

OCI APPLIED TO THE WAVE EQUATION

To solve (6.1)*

$$(3.1) \qquad u_{tt} = L_x(u) + L_y(u)$$

where initial and boundary data are prescribed and where

$$L_x(u) \equiv a u_{xx} + b u_x$$
$$L_y(u) \equiv c u_{yy} + d u_y$$

one substitutes the operator compact implicit approximation (2.4) for the respective spatial terms and the same compact implicit scheme as before for the temporal term to obtain

$$(3.2) \qquad Q_t^{-1} \frac{\delta_t^2}{k^2} U^n_{J,m} = \frac{(Q_x^n)^{-1} R_x^n}{h^2} U^n_{J,m} + \frac{(Q_y^n)^{-1} R_y^n}{h^2} U^n_{J,m}$$

where

$$(Q_x^n)^{-1} R_x^n U^n_{j,m} \sim L_x(u)^n_{j,m}$$
$$(Q_y^n)^{-1} R_y^n U_{j,m} \sim L_y(U)^n_{j,m}$$

[4] B. K. Swartz, "The Construction of Finite Difference Analogs of Some Finite Element Schemes," Mathematical Aspects of Finite Elements in Partial Differential Equations (C. DeBoor, Ed.) Academic Press, 1974, pp. 279-312.

Following our approach in [1] a factorization of (3.2) can be accomplished by adding the fourth order term

$$-\frac{k^4}{144}\,Q_t^{-1}\frac{\delta_t^2}{k^2}\,[(Q_x^n)^{-1}\,R_x^n\,(Q_y^n)^{-1}\,R_y^n]\,U_{j,m}^n$$

The resulting form (analogous to (2.9)*) is

$$(3.3)\qquad G_{j,m}^{n+1} \equiv [I - \frac{\lambda^2}{12}\,(Q_x^{n+1})^{-1}\,R_x^{n+1}][I - \frac{\lambda^2}{12}\,(Q_y^{n+1})^{-1}\,R_y^{n+1}]\,U_{j,m}^{n+1}$$

$$= 2G_{j,m}^n - G_{j,m}^{n-1} + \lambda^2[(Q_x^n)^{-1}\,R_x^n + (Q_y^n)^{-1}\,R_y^n]\,U_{j,m}^n$$

The numerical implementation of (3.3) above is completely analogous to the algorithm in [1] for equation (4.3)*. Again with

$$(3.4)\qquad [I - \frac{\lambda}{12}\,(Q_y^{n+1})^{-1}\,R_y^{n+1}]\,U_{j,m}^{n+1} \equiv Z_{j,m}^{n+1}$$

solve first for $Z_{j,m}^{n+1}$ and then for $U_{j,m}^{n+1}$.

The first two terms on the right hand side are known from previous time steps and need not be computed. The other two terms on the right hand side are obtained in a manner analogous to (4.5)*, (4.6)*. By observing that

$$(3.5)\qquad \lambda^2(Q_y^n)^{-1}\,R_y^n\,U_{j,m}^n = 12(U_{j,m}^n - Z_{j,m}^n)$$

the left hand side is obtained without much work. $V_{j,m}^n \equiv \lambda^2(Q_x^n)^{-1}\,R_x^n\,U_{j,m}^n$ is obtained by solving a tridiagonal system.

REMARKS

1. Here too, the algorithm for (3.3) requires that one generate initial data for $G_{j,m}^1$, $G_{j,m}^0$. These are obtained by solving tridiagonal systems as in (4.9)*.

2. Boundary Data: The computation of the intermediate boundary conditions may be obtained as in the previous paper. However, the following simplification is possible. As before at the four corner points use the analytic representation of $Z_{j,m}^{n+1}$ and one sided differences as an approximation for $Z_{j,m}^{n+1}$. Then on x = constant lines solve (3.4)

$$Q_y^{n+1}\,Z_{j,m}^{n+1} = [Q_y^{n+1} - \frac{\lambda^2}{12}\,R_y^{n+1}]U_{j,m}^{n+1}.$$

Observe, that the x-sweep part of the algorithm may also be implemented on y = constant boundary lines as well as on the interior lines. However, this leads to a problem. Experiments reveal that it is necessary to compute $Z^{n+1}_{j,m}$ on the y = constant boundaries as accurately as possible. Thus it is necessary to use the exact data for $U^{n+1}_{j,m}$ that is given on these lines. However, the definition of $G^{n+1}_{j,m}$ requires that

$$\lambda^2[(Q^n_x)^{-1} R^n_x + (Q^n_y)^{-1} R^n_y] U^n_{j,m},$$

be evaluated and $\lambda^2(Q^n_y)^{-1} R^n_y$ must be obtained from (3.5), which includes $Z^n_{j,m}$ which is not exact. However, this problem may be avoided by noting that

$$\lambda^2[(Q^n_x)^{-1} R^n_x + (Q^n_y)^{-1} R^n_y] U^n_{j,m} = \Delta t^2[L_x u^n + L_y u^n + O(h^4)] = \Delta t^2[u_{tt} + O(h^4)].$$

$G^{n+1}_{j,m}$ may then be redefined on y = constant <u>boundary lines</u> as

$$G^{n+1}_{j,m} = 2G^n_{j,m} - G^{n-1}_{j,m} + (I - \frac{\delta_t^2}{12})\delta_t^2 U^n_{j,m}$$

before implementing the first step of the algorithm.

A Fourier stability analysis of (3.3) reveals, by a perturbation argument, that so long as $r \sim h \sim k$ that again for $C = \max(a,b)$ and

$$\sqrt{C}\,\lambda \leq \sqrt{3} - 1$$

the amplification factors ρ satisfy

$$|\rho| \leq 1 + O(\Delta t)$$

and so the scheme is stable in the sense of von Neumann.

<u>Numerical Example</u>: In this section a numerical example is presented demonstrating the accuracy, effectiveness and the stability of the method.

Let Ω be defined by $[0 \leq x, y \leq .5]$ and define the coefficients, initial conditions, and boundary conditions of (3.1) by

$$a(x,y,t) = \frac{(x+1)^2}{4(t+1)^2}$$

$$b(x,y,t) = \frac{(x+1)^2(y+1)}{4(t+1)}$$

$$e(x,y,t) = \frac{(y+1)^2}{4(t+1)^2}$$

$$d(x,y,t) = \frac{(y+1)^2(x+1)}{4(t+1)}$$

$$u(x,y,o) = e^{(x+1)(y+1)}$$

$$u_t(x,y,o) = (x+1)(y+1)e^{(x+1)(y+1)}$$

$$u(x,y,t)|_{\partial\Omega} = e^{(x+1)(y+1)(t+1)}$$

The exact solution is

$$u(x,y,t) = e^{(x+1)(y+1)(t+1)}$$

The method was run for a sequence of spatial meshes and time steps, so that in each subsequent run we halved the mesh size and time step. In table 1 the accuracy of the method is demonstrated for L_2-error and relative max-error.

# Time Steps	h	k	L_2-Error	L_2-Rate	Relative Max-Error	Relative Max-Rate
20	.1	.05	3.847-05		5.049-06	
				3.74		3.91
40	.05	.025	2.886-06		3.341-07	
				3.97		3.98
80	.025	.0125	1.843-07		2.118-08	
				3.87		3.86
160	.0125	.00625	1.264-08		1.455-09	

Table 1

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